Technical Comments

Comment on "Modal Coupling in Lightly Damped Structures"

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HASSELMAN[1] has investigated criteria for modal coupling of lightly damped structures. His results are obtainable more simply and directly from the classical perturbation theory for lightly damped structures given by Rayleigh.[2] In addition, Rayleigh's derivation makes the mathematical treatment of orders of magnitude more explicit and obvious and permits computation of the frequency corrections for light damping to the second order in damping perturbation.

Rayleigh's theory begins with an unperturbed system that is undamped and is characterized by kinetic energy T and potential energy V in terms of the normal modes of vibration as generalized coordinates with amplitude q_n. These are, by definition, uncoupled. Thus,

$$T = \tfrac{1}{2}\, a_1 \dot{q}_1^2 + \tfrac{1}{2}\, a_2 \dot{q}_2^2 + \cdots \tag{1a}$$

$$V = \tfrac{1}{2}\, c_1 q_1^2 + c_2 q_2^2 + \cdots \tag{1b}$$

where, as is apparent, a_n and c_n are, respectively, the generalized mass and stiffness of the nth normal mode.

The perturbation comprises small damping terms that may couple the modes together. Thus damping is introduced in terms of the Rayleigh dissipation function F, where

$$F = \tfrac{1}{2}\, b_{11} \dot{q}_1^2 + \tfrac{1}{2}\, b_{22} \dot{q}_2^2 + \cdots + b_{12} \dot{q}_1 \dot{q}_2 + b_{13} \dot{q}_1 \dot{q}_3 + \cdots \tag{2}$$

Using Lagrange's equations with the dissipation terms and supposing variations of all time coordinates as e^{pt} leads to a set of equations equal in number to the number of degrees of freedom. The nth equation of this set is

$$[p^2 a_m + b_m p + c_m]\, q_m + p \Sigma b_{mn} q_n = 0 \tag{3}$$

Considering the damping as small and taking the unperturbed natural frequency $p_{k0} = i\omega_{k0}$, the first-order correction to the kth normal mode is obtained as the ratio of the nth modal coordinate to the kth:

$$\frac{q_n}{q_k} = -\frac{b_{kn} p_{k0}}{a_n (p_{n0}^2 - p_{k0}^2)} \tag{4}$$

This expression leads to criteria for modal coupling identical to those arrived at by Hasselman.

In addition, Rayleigh gives the equation for the computation of the natural frequencies (more properly in this case, eigenvalues that are complex for the damped case) to the second order in the magnitude of damping as

$$a_k p_k^2 + b_{kk} p_k + c_k + \sum_{n \neq k} \frac{p_{k0}^2 b_{kn}^2}{a_n (p_{n0}^2 - p_{k0}^2)} = 0 \tag{5}$$

Introducing the damping ratio of the kth mode as $\zeta_k = b_{kk}/2a_k\omega_{k0}$ and noting that $p_{k0} = i\omega_{k0}$, Eq. (4) becomes

$$\left| \frac{q_n}{q_k} \right| = \frac{2\,\zeta_k}{|(\omega_{k0}^2/\omega_{n0}^2) - 1|} \left(\frac{\xi_{kn}}{\xi_{kk}} \right) \tag{6}$$

where $\xi_{kk} = 2\zeta_k \omega_{k0}$ and $\xi_{kn} = b_{kn}/a_n$ to correspond to Hasselman's definitions. Here the purely imaginary phase of Eq. (4) has been suppressed by converting to absolute values. If, in Eq. (6), the requirement is set that $q_n/q_k \ll 1$, the result is exactly the inequality given by Hasselman's Eq. (15).

References

[1] Hasselman, T.K., "Modal Coupling in Lightly Damped Structures," *AIAA Journal*, Vol. 14, Nov. 1976, pp. 1627-1628.

[2] Rayleigh, B., *The Theory of Sound*, Dover, New York, 1945, Vol. 1, pp. 136-137.

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Reply by Author to A. H. Flax

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THE comment by Flax suggests an alternative criterion for neglecting the off-diagonal coupling terms in the modal damping matrix of a lightly damped structure. It first should be recognized as being different from the one presented in inequality (8) of Ref. 1. In particular, Flax implies that, whenever

$$|q_k/q_j| \ll 1 \tag{1}$$

the modal coupling term ξ_{kj} may be neglected. This inequality (1) leads to the criterion

$$\frac{2\zeta_j}{[(\beta^2-1)^2 + 4\zeta_k^2\beta^2]^{1/2}} \left| \frac{\xi_{kj}}{\xi_{jj}} \right| \ll 1 \tag{2}$$

whenever the kth equation (row) of the set

$$\{([\omega^2] - \Omega^2[I]) + i\Omega[\xi]\}\{q(i\Omega)\} = \{0\} \tag{3}$$

is used with $\Omega \equiv \omega_j$; $\beta = \omega_k/\omega_j$.

Reference 1 asserts that, whenever the inequality

$$|\{e\}_k^T [\tilde{Z}_n(i\Omega)]\{e\}_j| \ll 1 \tag{4}$$

is true, the modal coupling term ξ_{kj} may be neglected. This leads to the criterion

$$\left\{ \frac{2\zeta_j}{[(\beta^2-1)^2 + 4\zeta_k^2\beta^2]^{1/2}} \right\}^{1/2} \left| \frac{\xi_{kj}}{\xi_{jj}} \right| \ll 1 \tag{5}$$

which is clearly different from Flax's inequality (2).

The question is, which criterion (if either) is proper? If one were to choose $\Omega \equiv \omega_k$ in Eq. (3) for deriving the ratio q_k/q_j, inequality (1) would lead to

$$\frac{\zeta_j}{\zeta_k \beta} \left| \frac{\xi_{kj}}{\xi_{jj}} \right| \ll 1 \tag{6}$$

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which is a drastically different result. In the same manner, if one lets $\Omega=\omega_k$ in (4), the following results:

$$\left\{\frac{2\zeta_j}{[(\zeta_k/\zeta_j)^2(\beta^2-1)^2+4\zeta_k^2\beta^2]^{1/2}}\right\}^{1/2}\left|\frac{\xi_{kj}}{\xi_{jj}}\right|\ll 1 \tag{7}$$

Whenever $\zeta_j=\zeta_k$, the inequalities of (5) and (7) are identical. Otherwise they are not identical, and the statement immediately preceding Eq. (12) of Ref. 1 is not strictly true. Whenever $\zeta_j>\zeta_k$, assuming $\omega_j<\omega_k$, (7) should be applied in lieu of (5) (since the requirement of the latter is the more severe). The writer therefore is appreciative of this opportunity to modify the criterion offered in Ref. 1.

In any case, both of the inequalities (2) and (6), which follow from Flax's approach, are believed to be inappropriate. In the notation of Ref. 1,

$$[I+\tilde{Z}_n]H_\gamma=P_\gamma; \qquad H_\gamma=[I+\tilde{Z}_n]^{-1}P_\gamma$$

For small $\tilde{Z}_n$, it can be shown [Eq. (4A) of Ref. 2] that

$$[I+\tilde{Z}_n]^{-1}\approx[I-\tilde{Z}_n]$$

Transformation of H_γ to H_q yields

$$H_q=Z_d^{-1/2}[I-\tilde{Z}_n]Z_d^{-1/2}P_q \tag{8}$$

Recalling that $\tilde{Z}_n=Z_d^{-1/2}Z_nZ_d^{-1/2}$, we find that

$$H_q=(Z_d^{-1}-Z_d^{-1}Z_nZ_d^{-1})P_q \tag{9a}$$

$$H_q=(I-Z_d^{-1}Z_n)Z_d^{-1}P_q \tag{9b}$$

Flax's criterion is equivalent to the inequality

$$|e_k^TZ_d^{-1}(i\omega_j)Z_n(i\omega_j)e_j|\ll 1 \tag{10}$$

which may be inferred from Eq. (9). Contrast this with the inequality (4) derived from Eq. (8). If Flax's criterion (10) were to be used, it would seem that

$$|e_j^TZ_d^{-1}(i\omega_j)Z_n(i\omega_j)e_k|\ll 1 \tag{11}$$

also should be satisfied, recognizing the asymmetry of the matrix product $Z_d^{-1}Z_n$. This inequality (11) leads simply to

$$|\xi_{kj}/\xi_{jj}|\ll 1 \tag{12}$$

(assuming $\xi_{jk}=\xi_{kj}$), which takes us back to where we started, i.e., to the point of having to evaluate directly the importance of the off-diagonal terms in Eq. (3).

In conclusion, it is believed that Dr. Flax is in error with his statement that the writer's results "are directly obtainable from the classical perturbation theory...given by Rayleigh." The present reply serves to illustrate the differences between Flax's criterion and the one presented originally.[1]

References

[1] Hasselman, T.K., "Modal Coupling in Lightly Damped Structures," *AIAA Journal*, Vol. 14, Nov. 1976, pp. 1627-1628.

[2] Hasselman, T.K., "Damping Synthesis from Substructure Tests," *AIAA Journal*, Vol. 14, Oct. 1976, pp. 1409-1418.

Errata

Dynamical Constraints in Satellite Photogrammetry

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THE second line after Eq. (5c) should read: centered principal body axes system, denoted by the mutually

Equation (6c) should read:

$$\omega_3(t)=\omega_3[t,t_0,I_1,I_2,I_3,\omega_1(t_0),\omega_2(t_0),\omega_3(t_0)]$$

Equations (7b) and (7c) should read, respectively:

$$\phi_2(t)=\sin^{-1}\left(\frac{I_3\omega_3(t)}{H}\right), \quad -\frac{\pi}{2}\le\phi_2\le\frac{\pi}{2}$$

$$\phi_3(t)=\tan^{-1}\left(\frac{-I_2\omega_2(t)}{I_1\omega_1(t)}\right), \quad -\pi\le\phi_3\le\pi$$

Equation (12) should read:

$$[C(\omega(t),\phi(t),\kappa(t))]=[\Phi(t,t_0)][C(\omega(t_0),\phi(t_0),\kappa(t_0))]$$

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The fourth line from the bottom of the second column on page 490 should read: in order to test the feasibility of incorporating full dynamic

The nineteenth line on page 491 should read: University of Virginia CDC 6400 computer. These programs are

The fourth line in Appendix A should read: *Assume*: Principal body axes $\{\hat{p}\}$ are ordered such that

The eighth line after Eq. (A4) should read: Eqs. (A1) to uncouple the equations of motion. Equations (A1)

Equation (A6c) should read:

$$\left(\frac{dx}{d\tau}\right)^2=(1-x^2)(x^2-k^2)$$

The last line above Eq. (A9) should read: $s_i=+1, -1$, or 0, and

Equation (A10) should read:

$$|\omega_{3m}|=\left[\frac{2I_1T-H^2}{I_3(I_1-I_3)}\right]^{1/2}$$

Equation (A13b) should read:

$$=\left[\frac{(I_2-I_3)(2I_1T-H^2)}{I_1I_2I_3}\right]^{1/2} \text{ if } H^2<2I_2T$$

The sixth entry in the left-hand column of Table A1 should read: $\omega_3(t_0)\neq 0$

The eighth entry in the right-hand column of Table A1 should read: $s_3=s_1\,\mathrm{sign}(\omega_2(t_0))$

The second line in the title of Appendix B should read: Angles Orienting Principal Body Axes to an Angular